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The exact elasto-electric field of a rotating piezoceramic spherical shell with a functionally graded property

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Abstract

A displacement separation technique is employed to simplify the basic equations of a piezoceramic body with radial inhomogeneity. It is shown that the controlling equations are finally reduced to an uncoupled second-order ordinary differential equation and a coupled system of three second-order ordinary differential equations. Solutions to these differential equations are given for the case that material constants are of power functions of the radial coordinate. The static analysis of a steadily rotating spherical shell is then presented. © 2001 Elsevier Science Ltd. All rights reserved.

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1. Introduction

Piezoelectric materials (PZMs) nowadays have been widely employed as integrated structural elements in various applications. In practice, these smart components and elements are mostly fabricated in plate or shell configurations. Numerous investigations on the static and dynamic behaviors of piezoelectric plates and shells have been carried out (Batra and Liang, 1997; Bisegna and Maceri, 1996; Chen et al., 1997, 1998; Ding et al., 1997, 1999; Heyliger, 1997; Lee and Saravanos, 1997; Paul and Natarajan, 1996; Tzou and Zhong, 1994).

The most technologically important PZMs are poled ceramics that exhibit transverse isotropy with the unique axis aligned along the poling direction. When the ceramics are poled in the spherically radial direction, they will exhibit spherical isotropy, a special kind of transverse isotropy. Kirichok (1980) seemed to be the first to address the vibration problem of a piezoelectric sphere with spherical isotropy; however, only the simplest case, e.g. the purely radial vibration was considered. Shul'ga (1993) used separation formulae for displacements and shear stresses to analyze the general electroelastic oscillations of homogeneous spherical shells. Recently, Chen (1999) showed that the displacement separation technique developed for spherically isotropic elasticity (Ding and Chen, 1996; Chen, 1996) could be applied to PZMs and he

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presented exact solutions of some problems related to an infinite spherically isotropic piezoelastic medium. The method has been used to analyze a homogeneous piezoceramic hollow sphere rotating at a constant angular velocity (Chen and Ding, 1998).

Piezoelectric crystals besides being direction oriented could also exhibit inhomogeneity with reference to physical properties. Especially, the concept of functionally graded material (FGM) has been developed to optimize the response of structures. In FGMs, material constants usually vary along one or more directions continuously. Utilizing this inhomogeneity character, FGMs have found many important applications in from micro-electrical-mechanical systems (MEMS) to aerospace science. There are few theoretical works on piezoelectric plates or shells with functionally graded properties. Sarma (1980) has considered the torsional wave motion of an inhomogeneous piezoelectric cylindrical shell with finite length. Liu and Tani (1991, 1992) studied waves in piezoelectric plates with functionally graded properties by using a laminated approximation method. To the authors' knowledge, there is no study related to non-homogeneous piezoelectric spherical shells. It is noted here that Puro (1980) has applied a separation method to take account of the effect of radial inhomogeneity of a spherically isotropic purely elastic medium.

In this paper, the displacement separation technique (Ding and Chen, 1996; Chen and Ding, 1998; Chen, 1999) is further applied to simplify the basic equations of a spherically isotropic piezoelastic medium with radial inhomogeneity. Along with the function expansion method, the controlling equations are finally turned to an uncoupled second-order ordinary differential equation and a coupled system of three second-order ordinary differential equations. In the paper, attention will be paid to the case that the material constants are of power functions in the radial variable, of which, solutions to the resulting equations are derived. A steadily rotating piezoceramic spherical shell is then investigated and numerical results are given to show the effect of material inhomogeneity.

2. Basic formulations

In contrast to the work presented earlier for the homogeneous piezoelasticity (Chen and Ding, 1998; Chen, 1999), we here assume that all the material constants (five elastic constants, c_{ij}, two dielectric constants ε_{ij}, three piezoelectric constants, e_{ij}, and the mass density ρ) are functions of the radial coordinate r, i.e. the piezoelectric medium under consideration is non-homogeneous along the radial direction.

For the analysis, the following displacement separation technique is valid as for the homogeneous piezoelasticity:

$$u_\theta = -\frac{1}{\sin\theta}\frac{\partial\psi}{\partial\phi} - \frac{\partial G}{\partial\theta}, \quad u_\phi = \frac{\partial\psi}{\partial\theta} - \frac{1}{\sin\theta}\frac{\partial G}{\partial\phi}, \quad u_r = w. \tag{1}$$

It is also assumed that the body force components F_i $(i = r, \theta, \phi)$ can be decomposed in the same way,

$$rF_\theta = -\frac{1}{\sin\theta}\frac{\partial V}{\partial\phi} - \frac{\partial U}{\partial\theta}, \quad rF_\phi = \frac{\partial V}{\partial\theta} - \frac{1}{\sin\theta}\frac{\partial U}{\partial\phi}, \quad rF_r = W. \tag{2}$$

By employing Eqs. (1) and (2), through some lengthy manipulations, we can transfer the equations of equilibrium (Love, 1927) and the charge equation of electrostatics (Tiersten, 1969; Shul'ga, 1993; Chen, 1999) to the following equations:

$$A + (\nabla_2 c_{44})w - (\nabla_2 c_{44})(\nabla_2 G - G) + (\nabla_2 e_{15})\Phi - rU = 0, \tag{3}$$

$$B + (\nabla_2 c_{44})(\nabla_2\psi - \psi) + rV = 0. \tag{4}$$

$$[L_3 + 2(\nabla_2 c_{13}) + (\nabla_2 c_{33})\nabla_2]w - [L_4 + (\nabla_2 c_{13})]\nabla_1^2 G + [L_5 + (\nabla_2 e_{33})\nabla_2]\Phi + rW = 0, \tag{5}$$

$$[L_7 + 2(\nabla_2 e_{31}) + (\nabla_2 e_{33})\nabla_2]w - [L_8 + (\nabla_2 e_{31})]\nabla_1^2 G - [L_9 + (\nabla_2 \varepsilon_{33})\nabla_2]\Phi = r^2 \rho_{\mathrm{f}}, \tag{6}$$

where ρ_{f} is the free charge density in the piezoelastic body, and

$$\begin{aligned}
&A = L_1 w - L_2 G + L_6 \Phi, \qquad B = [c_{44}\nabla_3^2 - 2c_{44} + c_{11} - c_{12} + \tfrac{1}{2}(c_{11} - c_{12})\nabla_1^2]\psi,\\
&L_1 = (c_{13} + c_{44})\nabla_2 + c_{11} + 2c_{44} + c_{12}, \qquad L_2 = c_{44}\nabla_3^2 - 2c_{44} + c_{11} - c_{12} + c_{11}\nabla_1^2,\\
&L_3 = c_{33}\nabla_3^2 - 2(c_{11} + c_{12} - c_{13}) + c_{44}\nabla_1^2, \qquad L_4 = (c_{13} + c_{44})\nabla_2 - c_{44} - c_{11} - c_{12} + c_{13},\\
&L_5 = e_{33}\nabla_3^2 - 2e_{31}\nabla_2 + e_{15}\nabla_1^2, \qquad L_6 = (e_{15} + e_{31})\nabla_2 + 2e_{15},\\
&L_7 = e_{33}\nabla_3^2 + 2e_{31}\nabla_2 + 2e_{31} + e_{15}\nabla_1^2, \qquad L_8 = (e_{31} + e_{15})\nabla_2 + e_{31} - e_{15},\\
&L_9 = \varepsilon_{33}\nabla_3^2 + \varepsilon_{11}\nabla_1^2, \qquad \nabla_2 = r\frac{\partial}{\partial r}, \qquad \nabla_2^2 = r\frac{\partial}{\partial r} r\frac{\partial}{\partial r}, \qquad \nabla_3^2 = \nabla_2^2 + \nabla_2,\\
&\nabla_1^2 = \frac{\partial^2}{\partial \theta^2} + \cot\theta\frac{\partial}{\partial\theta} + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}.
\end{aligned} \tag{7}$$

Thus, we turn the basic equations to Eqs. (3)–(6), from which, we find function ψ is uncoupled from the other two displacement functions w and G, and the electric potential Φ . In particular, Eq. (4) is a second-order uncoupled partial differential equation in ψ; Eqs. (3), (5) and (6) form a coupled partial differential equation system in w, G and Φ.

For closed spherical shells, noticing that the resulting equations include the partial operator ∇_1^2, which is defined in Eq. (7), it is assumed that

$$\begin{cases}
\psi = R\sum_{n=1}^{\infty}\psi_n(\xi)S_n^m(\theta,\phi), & w = R\sum_{n=0}^{\infty}w_n(\xi)S_n^m(\theta,\phi),\\
G = R\sum_{n=1}^{\infty}G_n(\xi)S_n^m(\theta,\phi), & \Phi = R\frac{e_{33}}{\varepsilon_{33}}\sum_{n=0}^{\infty}\Phi_n(\xi)S_n^m(\theta,\phi),
\end{cases} \tag{8}$$

and

$$\begin{cases}
V = -c_{44}\sum_{n=1}^{\infty}V_n(\xi)S_n^m(\theta,\phi), & U = -c_{44}\sum_{n=1}^{\infty}U_n(\xi)S_n^m(\theta,\phi),\\
W = -c_{33}\sum_{n=0}^{\infty}W_n(\xi)S_n^m(\theta,\phi), & r\rho_{\mathrm{f}} = -e_{33}\sum_{n=0}^{\infty}X_n(\xi)S_n^m(\theta,\phi),
\end{cases} \tag{9}$$

where $\xi = r/R$ is the non-dimensional radial coordinate, R is the mean radius of the shell, and $S_n^m(\theta,\phi)$ are spherical harmonics. Substituting Eqs. (8) and (9) into Eqs. (3)–(6), yields

$$\xi^2\psi_n'' + (f_9 + 2)\xi\psi_n' - [2 + (n^2 + n - 2)(f_1 - f_2)/2 + f_9]\psi_n = \xi V_n, \tag{10}$$

$$\begin{aligned}
&\xi^2 w_n'' + (f_{10} + 2)\xi w_n' + (p_1 + 2f_{11})w_n - p_2\xi G_n' - [p_3 - n(n+1)f_{11}]G_n + q_1\xi^2\Phi_n''\\
&\quad + (q_2 + f_8 f_{12}/f_4)\xi\Phi_n' + q_3\Phi_n = \xi W_n,
\end{aligned} \tag{11}$$

$$\xi^2 G_n'' + (f_9 + 2)\xi G_n' + (p_4 - f_9)G_n - p_5\xi w_n' - (p_6 + f_9)w_n + q_4\xi\Phi_n' + (q_5 + f_{13})\Phi_n = \xi U_n, \tag{12}$$

$$\begin{aligned}
&\xi^2\Phi_n'' + (f_{14} + 2)\xi\Phi_n' + q_6\Phi_n - \xi^2 w_n'' - (p_7 + f_{12})\xi w_n' - (p_8 + f_{15})w_n - p_9\xi G_n' - (p_{10} + f_{16})G_n\\
&\quad = \xi X_n,
\end{aligned} \tag{13}$$

where a prime denotes differentiation with respect to ξ, and

$$\begin{cases} p_1 = [2(f_3 - f_1 - f_2) - n(n+1)]/f_4, \quad p_2 = -n(n+1)(f_3+1)/f_4, \\ p_3 = n(n+1)(f_1 + f_2 + 1 - f_3)/f_4, \quad p_4 = f_1 - f_2 - n(n+1)f_1 - 2, \\ p_5 = f_3 + 1, \; p_6 = f_1 + f_2 + 2, \quad p_7 = 2(f_6 + 1), \quad p_8 = 2f_6 - n(n+1)f_5, \\ p_9 = n(n+1)(f_5 + f_6), \quad p_{10} = n(n+1)(f_6 - f_5), \quad q_1 = f_8/f_4, \\ q_2 = 2f_8(1 - f_6)/f_4, \quad q_3 = -n(n+1)f_5 f_8/f_4, \quad q_4 = -(f_5 + f_6)f_8, \\ q_5 = -2f_5 f_8, \quad q_6 = -n(n+1)f_7, \\ f_1 = c_{11}/c_{44}, \quad f_2 = c_{12}/c_{44}, \quad f_3 = c_{13}/c_{44}, \quad f_4 = c_{33}/c_{44}, \\ f_5 = e_{15}/e_{33}, \quad f_6 = e_{31}/e_{33}, \quad f_7 = \varepsilon_{11}/\varepsilon_{33}, \quad f_8 = e_{33}^2/(\varepsilon_{33}c_{44}), \\ f_9 = (\nabla_2 c_{44})/c_{44}, \quad f_{10} = (\nabla_2 c_{33})/c_{33}, \quad f_{11} = (\nabla_2 c_{13})/c_{33}, \\ f_{12} = (\nabla_2 e_{33})/e_{33}, \quad f_{13} = -f_8(\nabla_2 e_{15})/e_{33}, \quad f_{14} = (\nabla_2 \varepsilon_{33})/\varepsilon_{33}, \\ f_{15} = 2(\nabla_2 e_{31})/e_{33}, \quad f_{16} = n(n+1)(\nabla_2 e_{31})/e_{33}. \end{cases} \tag{14}$$

It is obvious that Eq. (10) is an independent second-order ordinary differential equation in the unknown ψ_n. Eqs. (11)–(13) are coupled by the three unknowns G_n, w_n and Φ_n, and each equation involved is a second-order ordinary differential one. In the next section, solutions to Eqs. (10)–(13) will be given for the particular case that all material constants are of power functions with an identical exponent of the radial coordinate.

3. Solutions

We assume here that the distributions of material constants obey a same power law along the radial direction, i.e. $c_{ij} = c_{ij}^0 \xi^\alpha$, $e_{ij} = e_{ij}^0 \xi^\alpha$, $\varepsilon_{ij} = \varepsilon_{ij}^0 \xi^\alpha$ and $\rho = \rho_0 \xi^\alpha$, here c_{ij}^0, e_{ij}^0, ε_{ij}^0 and ρ_0 are constants. Eqs. (10)–(14) keep unaltered except that the following non-dimensional parameters read

$$f_9 = f_{10} = f_{12} = f_{14} = \alpha, \quad f_{11} = \alpha f_3/f_4, \quad f_{13} = -\alpha f_5 f_8, \quad f_{15} = 2\alpha f_6, \quad f_{16} = n(n+1)\alpha f_6. \tag{15}$$

It is noted here that the non-dimensional parameters f_i $(i = 1, 2, \ldots, 8)$ defined in Eq. (14) now take forms such as $f_1 = c_{11}^0/c_{44}^0$ and $f_2 = c_{12}^0/c_{44}^0$, etc.

3.1. The general solution

It can be seen that the corresponding homogeneous Eqs. (10)–(13) are of the Euler type so that the general solution (or homogeneous solution) can be obtained by assuming:

$$G_n = A_n \xi^{\nu_n - (1+\alpha)/2}, \quad w_n = B_n \xi^{\nu_n - (1+\alpha)/2}, \quad \Phi_n = C_n \xi^{\nu_n - (1+\alpha)/2}, \quad \psi_n = D_n \xi^{\lambda_n - (1+\alpha)/2}, \tag{16}$$

where A_n, B_n, C_n, and D_n are undetermined constants. Substituting Eq. (16) into Eqs. (10)–(13) and omitting the right-hand sides, yields

$$\{\lambda_n^2 - \tfrac{1}{4}[9 + 2(n-1)(n+2)(f_1 - f_2) + \alpha(6+\alpha)]\}D_n = 0, \tag{17}$$

and

$$\mathbf{H}(\nu_n)\begin{Bmatrix} A_n \\ B_n \\ C_n \end{Bmatrix} = 0, \tag{18}$$

with

$$\mathbf{H}(\nu_n) = \begin{bmatrix} \nu_n^2 - (1+\alpha)^2/4 + p_4 - \alpha & -p_5[\nu_n - (1+\alpha)/2] - p_6 - \alpha & q_4[\nu_n - (1+\alpha)/2] + q_5 + f_{13} \\ -p_2[\nu_n - (1+\alpha)/2] - p_3 + n(n+1)f_{11} & \nu_n^2 - (1+\alpha)^2/4 + p_1 + 2f_{11} & q_1\{\nu_n^2 - (1+\alpha)^2/4 - 2f_6[\nu_n - (1+\alpha)/2]\} + q_3 \\ -p_9[\nu_n - (1+\alpha)/2] - p_{10} - f_{16} & -\nu_n^2 + (1+\alpha)^2/4 - 2f_6[\nu_n - (1+\alpha)/2] - p_8 - f_{15} & \nu_n^2 - (1+\alpha)^2/4 + q_6 \end{bmatrix}. \tag{19}$$

For non-zero D_n, Eq. (17) gives the eigenvalues of λ_n as follows

$$\lambda_{n1,2} = \pm\tfrac{1}{2}[9 + 2(n-1)(n+2)(f_1 - f_2) + \alpha(6+\alpha)]^{1/2}. \tag{20}$$

From Eq. (18), the eigenequation determining v_n is obtained

$$|\mathbf{H}(v_n)| = 0. \tag{21}$$

As the homogeneous case (Chen and Ding, 1998), Eq. (21) is a cubic algebraic equation in v_n^2. For stable materials, the eigenvalue v_n cannot be purely imaginary. We assume that $v_{ni} = -v_{n(i+3)}$ with $\mathrm{Re}[v_{ni}] < 0$, $(i = 1, 2, 3)$ and that the six eigenvalues are distinct. One can then obtain the following relationships from Eq. (18)

$$B_{ni} = K_{ni}^1 A_{ni}, \qquad C_{ni} = K_{ni}^2 A_{ni}, \tag{22}$$

for each eigenvalue v_{ni}, $(i = 1, 2, \ldots, 6)$, where K_{ni}^1 and K_{ni}^2 are solved from two independent equations in Eq. (18). Based on the above results, we obtain a general solution as follows:

$$\begin{aligned}
u_r &= R\sum_{n=0}^{\infty}\sum_{i=1}^{6} K_{ni}^1 A_{ni}\xi^{v_{ni}-(1+\alpha)/2} S_n^m(\theta,\phi),\\
u_\theta &= -R\sum_{n=1}^{\infty}\sum_{i=1}^{6} A_{ni}\xi^{v_{ni}-(1+\alpha)/2}\frac{\partial}{\partial\theta}S_n^m(\theta,\phi) - R\sum_{n=1}^{\infty}\sum_{i=1}^{2} D_{ni}\xi^{\lambda_{ni}-(1+\alpha)/2}\frac{1}{\sin\theta}\frac{\partial}{\partial\phi}S_n^m(\theta,\phi),\\
u_\phi &= -R\sum_{n=1}^{\infty}\sum_{i=1}^{6} A_{ni}\xi^{v_{ni}-(1+\alpha)/2}\frac{1}{\sin\theta}\frac{\partial}{\partial\phi}S_n^m(\theta,\phi) + R\sum_{n=1}^{\infty}\sum_{i=1}^{2} D_{ni}\xi^{\lambda_{ni}-(1+\alpha)/2}\frac{\partial}{\partial\theta}S_n^m(\theta,\phi),\\
\Phi &= R\frac{e_{33}}{\varepsilon_{33}}\sum_{n=0}^{\infty}\sum_{i=1}^{6} K_{ni}^2 A_{ni}\xi^{v_{ni}-(1+\alpha)/2} S_n^m(\theta,\phi).
\end{aligned} \tag{23}$$

It should be noted that Eqs. (10)–(13) degenerate to the following two Eqs. when $n = 0$:

$$\xi^2 w_0'' + (2+\alpha)\xi w_0' + (p_1 + 2f_{11})w_0 + q_1\xi^2\Phi_0'' + (q_2 + f_8 f_{12}/f_4)\xi\Phi_0' = \xi W_0, \tag{24}$$

$$\xi^2\Phi_0'' + (2+\alpha)\xi\Phi_0' - \xi^2 w_0'' - (p_7+\alpha)\xi w_0' - (p_8 + f_{15})w_0 = \xi X_0. \tag{25}$$

Consequently, the following fourth-order eigenequation is derived:

$$\begin{aligned}
|N(v_0)| &= \begin{vmatrix} q_1\{v_0^2 - \frac{1}{4}(1+\alpha)^2 - 2f_6[v_0 - \frac{1}{2}(1+\alpha)]\} & v_0^2 - \frac{1}{4}(1+\alpha)^2 + p_1 + 2f_{11} \\ v_0^2 - \frac{1}{4}(1+\alpha)^2 & -v_0^2 + \frac{1}{4}(1+\alpha)^2 - 2f_6[v_0 - \frac{1}{2}(1+\alpha)] - p_8 - f_{15} \end{vmatrix}\\
&= -[v_0^2 - \tfrac{1}{4}(1+\alpha)^2]\{(1+q_1)[v_0^2 - \tfrac{1}{4}(1+\alpha)^2] - 2f_6 q_1(2f_6 - 1 - \alpha) + p_1 + 2f_{11}\} = 0.
\end{aligned} \tag{26}$$

Thus to write in a united form as given in Eq. (23), we shall employ the following formulae for $n = 0$ there:

$$A_{0i} = C_{0i}, \qquad K_{0i}^2 = 1, \qquad K_{0i}^1 = -\frac{q_1\{v_{0i}^2 - (1+\alpha)^2/4 - 2f_6[v_{0i} - (1+\alpha)/2]\}}{v_{0i}^2 - (1+\alpha)/4 + p_1 + 2f_{11}} \quad (i = 1, 2, 4, 5) \tag{27}$$

and $A_{03} = A_{06} = 0$. It can be shown that the terms in the general solution corresponding to $v_{04} = (1+\alpha)/2$, $v_{14} = (1+\alpha)/2$ and $\lambda_{12} = (3+\alpha)/2$ all give zero stress and electric displacement fields. Without the loss of generality, we assume A_{04}, A_{14} and D_{12} to be zero.

Setting $\alpha = 0$ and neglecting the piezoelectric effect, expressions obtained in Eq. (23) degenerate identically to those of the homogeneous pure elasticity (Chen, 1966).

3.2. The particular solution

Similar to the homogeneous pure elasticity (Chen, 1966), we assume the body forces and the free charge density to be in the following form:

$$\begin{aligned} F_r &= \sum_{n=0}^{\infty} E_n \xi^{\mu_n-(5-\alpha)/2} S_n^m(\theta,\phi), \quad F_\theta = \sum_{n=1}^{\infty} F_n \xi^{\mu_n-(5-\alpha)/2} \frac{\partial}{\partial\theta} S_n^m(\theta,\phi), \\ F_\phi &= \sum_{n=1}^{\infty} F_n \xi^{\mu_n-(5-\alpha)/2} \frac{1}{\sin\theta}\frac{\partial}{\partial\phi} S_n^m(\theta,\phi), \quad \rho_{\mathrm{f}} = \sum_{n=1}^{\infty} H_n \xi^{\mu_n-(5-\alpha)/2} S_n^m(\theta,\phi), \end{aligned} \tag{28}$$

where E_n, F_n, H_n and μ_n are known constants. From Eqs. (2), (9) and (28), one obtains

$$\begin{aligned} &V_n(\xi) = 0, \quad U_n(\xi) = (RF_n/c_{44}^0)\xi^{\mu_n-(3+\alpha)/2} \\ &W_n(\xi) = -(RE_n/c_{33}^0)\xi^{\mu_n-(3+\alpha)/2}, \quad X_n(\xi) = -(RH_n/e_{33}^0)\xi^{\mu_n-(3+\alpha)/2}. \end{aligned} \tag{29}$$

To find the particular solution to the inhomogeneous Eqs. (10)–(13) with the body forces and the free charge density as given in Eq. (28), it is assumed that

$$\begin{aligned} u_\theta^* &= -R\sum_{n=1}^{\infty} A_n^* \xi^{\mu_n-(1+\alpha)/2} \frac{\partial}{\partial\theta} S_n^m(\theta,\phi), \quad u_\phi^* = -R\sum_{n=1}^{\infty} A_n^* \xi^{\mu_n-(1+\alpha)/2} \frac{1}{\sin\theta}\frac{\partial}{\partial\phi} S_n^m(\theta,\phi), \\ u_r^* &= R\sum_{n=0}^{\infty} B_n^* \xi^{\mu_n-(1+\alpha)/2} S_n^m(\theta,\phi), \quad \Phi^* = R\frac{e_{33}^0}{\varepsilon_{33}^0}\sum_{n=0}^{\infty} C_n^* \xi^{\mu_n-(1+\alpha)/2} S_n^m(\theta,\phi). \end{aligned} \tag{30}$$

The equations to determine the constants A_n^*, B_n^*, and C_n^* are thus obtained from Eqs. (11)–(13) or Eqs. (24) and (25):

$$\mathbf{H}(\mu_n)\begin{Bmatrix} A_n^* \\ B_n^* \\ C_n^* \end{Bmatrix} = \begin{Bmatrix} RF_n/c_{44}^0 \\ -RE_n/c_{33}^0 \\ -RH_n/e_{33}^0 \end{Bmatrix}, \tag{31}$$

for $n > 0$, and

$$\mathbf{N}(\mu_0)\begin{Bmatrix} C_0^* \\ B_0^* \end{Bmatrix} = -\begin{Bmatrix} RE_0/c_{33}^0 \\ RH_0/e_{33}^0 \end{Bmatrix}, \tag{32}$$

for $n = 0$.

Obviously, the complete solution includes two different contributions, i.e. the homogeneous solution (23) and the particular solution (30). In the next section, we shall consider the general axisymmetric boundary-value problem of a piezoceramic spherical shell.

4. Axisymmetric boundary-value problem

We shall pay attention to the axisymmetric boundary-value problem of a piezoelectric spherical shell with the inner radius a and the outer radius b, in absence of body forces and free charge density. In the axisymmetric case, Eq. (23) becomes

$$u_r = R\sum_{n=0}^{\infty}\sum_{i=1}^{6}K_{ni}^1 A_{ni}\xi^{\nu_{ni}-(1+\alpha)/2}P_n(\cos\theta), \quad u_\theta = -R\sum_{n=1}^{\infty}\sum_{i=1}^{6}A_{ni}\xi^{\nu_{ni}-(1+\alpha)/2}\frac{\partial}{\partial\theta}P_n(\cos\theta),$$
$$u_\phi = R\sum_{n=1}^{\infty}\sum_{i=1}^{2}D_{ni}\xi^{\lambda_{ni}-(1+\alpha)/2}\frac{\partial}{\partial\theta}P_n(\cos\theta), \quad \Phi = R\frac{e_{33}}{\varepsilon_{33}}\sum_{n=0}^{\infty}\sum_{i=1}^{6}K_{ni}^2 A_{ni}\xi^{\nu_{ni}-(1+\alpha)/2}P_n(\cos\theta), \tag{33}$$

where $P_n(\cos\theta)$ are Legendre polynomials. The stress and electric displacement components are:

$$\begin{aligned}
\sigma_{rr}^g &= \sum_{n=0}^{\infty}\sum_{i=1}^{6}A_{ni}\left\{c_{33}^0 K_{ni}^1[\nu_{ni}-(1+\alpha)/2] + c_{13}^0[2K_{ni}^1 + n(n+1)] \right.\\
&\quad \left. + \frac{(e_{33}^0)^2}{\varepsilon_{33}^0}K_{ni}^2[\nu_{ni}-(1+\alpha)/2]\right\}\xi^{\nu_{ni}-(3-\alpha)/2}P_n(\cos\theta),\\
\sigma_{r\theta}^g &= \sum_{n=1}^{\infty}\sum_{i=1}^{6}A_{ni}\left\{c_{44}^0[K_{ni}^1 - \nu_{ni} + (3+\alpha)/2] + \frac{e_{15}^0 e_{33}^0}{\varepsilon_{33}^0}K_{ni}^2\right\}\xi^{\nu_{ni}-(3-\alpha)/2}\frac{\partial}{\partial\theta}P_n(\cos\theta),\\
\sigma_{\theta\theta}^g + \sigma_{\phi\phi}^g &= \sum_{n=0}^{\infty}\sum_{i=1}^{6}A_{ni}\left\{(c_{11}^0 + c_{12}^0)[2K_{ni}^1 + n(n+1)] + 2c_{13}^0 K_{ni}^1[\nu_{ni}-(1+\alpha)/2] \right.\\
&\quad \left. + \frac{2e_{31}^0 e_{33}^0}{\varepsilon_{33}^0}K_{ni}^2[\nu_{ni}-(1+\alpha)/2]\right\}\xi^{\nu_{ni}-(3-\alpha)/2}P_n(\cos\theta),\\
\sigma_{\theta\theta}^g - \sigma_{\phi\phi}^g &= \sum_{n=1}^{\infty}\sum_{i=1}^{6}A_{ni}(c_{11}^0 - c_{12}^0)\left[n(n+1)P_n(\cos\theta) + 2\cot\theta\frac{\partial}{\partial\theta}P_n(\cos\theta)\right]\xi^{\nu_{ni}-(3-\alpha)/2},\\
D_r^g &= \sum_{n=0}^{\infty}\sum_{i=1}^{6}A_{ni}\{e_{33}^0 K_{ni}^1[\nu_{ni}-(1+\alpha)/2] + e_{31}^0[2K_{ni}^1 + n(n+1)]\\
&\quad - \varepsilon_{33}^0 K_{ni}^2[\nu_{ni}-(1+\alpha)/2]\}\xi^{\nu_{ni}-(3-\alpha)/2}P_n(\cos\theta),\\
D_\theta^g &= \sum_{n=1}^{\infty}\sum_{i=1}^{6}A_{ni}\left\{e_{15}^0[K_{ni}^1 - \nu_{ni} + (3+\alpha)/2] - \frac{\varepsilon_{11}^0 e_{33}^0}{\varepsilon_{33}^0}K_{ni}^2\right\}\xi^{\nu_{ni}-(3-\alpha)/2}\frac{\partial}{\partial\theta}P_n(\cos\theta),
\end{aligned} \tag{34}$$

Here the superscript g indicates the variable corresponding to the general solution. On the inner and outer spherical surfaces $r = a$ and $r = b$, it is assumed that the surface tractions and electric charge are sufficiently smooth to admit the representations (Chen, 1966)

$$\begin{aligned}
\sigma_{rr}(\xi_j) &= \sum_{n=0}^{\infty}\zeta_n^{(j)}P_n(\cos\theta),\\
\sigma_{r\theta}(\xi_j) &= \sum_{n=1}^{\infty}\eta_n^{(j)}\frac{\partial}{\partial\theta}P_n(\cos\theta),\\
\sigma_{r\phi}(\xi_j) &= \sum_{n=1}^{\infty}\tau_n^{(j)}\frac{\partial}{\partial\theta}P_n(\cos\theta),\\
D_r(\xi_j) &= \sum_{n=0}^{\infty}\kappa_n^{(j)}P_n(\cos\theta),
\end{aligned} \tag{35}$$

where $j = 1,2$, $\xi_1 = a/R$ and $\xi_2 = b/R$ are the two dimensionless radii, and the coefficients $\zeta_n^{(j)}$, $\eta_n^{(j)}$, $\tau_n^{(j)}$, and $\kappa_n^{(j)}$ are known for all n. They can be found through the orthogonality properties of Legendre polynomials.

In order that the shell is in mechanical equilibrium, it is necessary that the z-components of the sum of the resultant forces and moments on the $\xi_1 = a/R$ and $\xi_2 = b/R$ boundaries be zero, which leads to

$$\left[\zeta_1^{(1)} - 2\eta_1^{(1)}\right]\xi_1^2 = \left[\zeta_1^{(2)} - 2\eta_1^{(2)}\right]\xi_2^2, \tag{36}$$

$$\tau_1^{(1)}\xi_1^3 - \tau_2^{(2)}\xi_2^3 = 0. \tag{37}$$

Moreover, the electric equilibrium condition demands

$$\kappa_0^{(1)}\xi_1^2 - \kappa_0^{(2)}\xi_2^2 = 0. \tag{38}$$

The arbitrary constants A_{nj} $(j = 1, 2, \ldots, 6)$ are completely determined by Eqs. (36) and (38), together with the following sets of equations, which are obtained by comparing coefficient of Eqs. (34) and (35):

$n = 0$

$$\sum_{i=1}^{6} A_{0i}\left[c_{33}^0 K_{0i}^1\left(v_{0i} - \frac{1+\alpha}{2}\right) + 2c_{13}^0 K_{0i}^1 + \frac{(e_{33}^0)^2}{\varepsilon_{33}^0} K_{0i}^2\left(v_{0i} - \frac{1+\alpha}{2}\right)\right]\xi_j^{v_{0i}-(3-\alpha)/2} = \zeta_0^{(j)} \quad (j = 1, 2), \tag{39}$$

$$\sum_{i=1}^{6} A_{0i}\left[e_{33}^0 K_{0i}^1\left(v_{0i} - \frac{1+\alpha}{2}\right) + 2e_{31}^0 K_{0i}^1 - e_{33}^0 K_{0i}^2\left(v_{0i} - \frac{1+\alpha}{2}\right)\right]\xi_j^{v_{0i}-(3-\alpha)/2} = \kappa_0^{(j)} \quad (j = 1, 2), \tag{40}$$

$n > 0$

$$\sum_{i=1}^{6} A_{ni}\left\{c_{33}^0 K_{ni}^1\left(v_{ni} - \frac{1+\alpha}{2}\right) + c_{13}^0[2K_{ni}^1 + n(n+1)] + \frac{(e_{33}^0)^2}{\varepsilon_{33}^0} K_{ni}^2\left(v_{ni} - \frac{1+\alpha}{2}\right)\right\}\xi_j^{v_{ni}-(3-\alpha)/2} = \zeta_n^{(j)} \quad (j = 1, 2), \tag{41}$$

$$\sum_{i=1}^{6} A_{ni}\left\{e_{33}^0 K_{ni}^1\left(v_{ni} - \frac{1+\alpha}{2}\right) + e_{31}^0[2K_{ni}^1 + n(n+1)] - e_{33}^0 K_{ni}^2\left(v_{ni} - \frac{1+\alpha}{2}\right)\right\}\xi_j^{v_{ni}-(3-\alpha)/2} = \kappa_n^{(j)} \quad (j = 1, 2), \tag{42}$$

$$\sum_{i=1}^{6} A_{ni}\left\{c_{44}^0\left[K_{ni}^1 - \left(v_{ni} - \frac{3+\alpha}{2}\right)\right] + \frac{e_{15}^0 e_{33}^0}{\varepsilon_{33}^0} K_{ni}^2\right\}\xi_j^{v_{ni}-(3-\alpha)/2} = \eta_n^{(j)} \quad (j = 1, 2), \tag{43}$$

$$A_{03} = A_{04} = A_{06} = A_{14} = 0. \tag{44}$$

Similarly, the constants D_{n1} and D_{n2} are determined by Eq. (37) and the following set of equations:

$$\sum_{i=1}^{2} c_{44}^0 D_{ni}\left(\lambda_{ni} - \frac{3+\alpha}{2}\right)\xi_j^{\lambda_{ni}-(3-\alpha)/2} = \tau_n^{(j)} \quad (j = 1, 2), \tag{45}$$

$$D_{12} = 0. \tag{46}$$

It is seen that when $n = 0$, there are just three unknowns A_{01}, A_{02} and A_{05}. However, we have totally four equations in Eqs. (39) and (40). By virtue of Eqs. (26) and (27), it can be verified for $n = 0$ that,

$$
\begin{aligned}
&e_{33}^0 K_{0i}^1\left(v_{0i}-\frac{1+\alpha}{2}\right)+2e_{31}^0K_{0i}^1-e_{33}^0K_{0i}^2\left(v_{0i}-\frac{1+\alpha}{2}\right)\\
&=-\left[e_{33}^0\left(v_{0i}-\frac{1+\alpha}{2}\right)+2e_{31}^0\right]\frac{q_1\left[v_{0i}^2-\frac{(1+\alpha)^2}{4}\right]-2f_6q_1\left(v_{0i}-\frac{1+\alpha}{2}\right)}{v_{0i}^2-(1+\alpha)^2/4+p_1+2f_{11}}-e_{33}^0\left(v_{0i}-\frac{1+\alpha}{2}\right)\\
&=-e_{33}^0\left(v_{0i}-\frac{1+\alpha}{2}\right)\left\{\frac{q_1\left[v_{0i}^2-\frac{(1+\alpha)^2}{4}\right]+2f_6q_1(1+\alpha)-4f_6^2q_1}{v_{0i}^2-(1+\alpha)^2/4+p_1+2f_{11}}+1\right\}\\
&=-e_{33}^0\left(v_{0i}-\frac{1+\alpha}{2}\right)\frac{(1+q_1)\left[v_{0i}^2-\frac{(1+\alpha)^2}{4}\right]-2f_6q_1(2f_6-1-\alpha)+p_1+2f_{11}}{v_{0i}^2-(1+\alpha)^2/4+p_1+2f_{11}}=0\quad(i=2,5).
\end{aligned}
\tag{47}
$$

Since $v_{01}=-(1+\alpha)/2$, one finds that Eq. (40), which now reads as

$$
A_{01}[-e_{33}^0K_{01}^1(1+\alpha)+2e_{31}^0K_{01}^1+e_{33}^0(1+\alpha)]=\kappa_0^{(j)}\xi_j^2 \tag{48}
$$

is compatible with the electric equilibrium condition (38). Thus from Eqs. (39) and (48), we can uniquely determine the three unknowns A_{01}, A_{02} and A_{05}. A similar but a little more complicated verification can also be given in the case of $n=1$, which is left to the interested reader.

5. A rotating piezoceramic spherical shell

The problem of a rotating spherical shell also can be treated as an axisymmetric boundary-value problem. The non-trivial stresses and electric displacements, however, should involve the terms corresponding to the particular solution:

$$
\begin{aligned}
\sigma_{rr}&=\sigma_{rr}^g+\sum_{n=0}^{\infty}A_n^*\left\{c_{33}^0L_n^1[\mu_n-(1+\alpha)/2]+c_{13}^0[2L_n^1+n(n+1)]\right.\\
&\quad\left.+\frac{(e_{33}^0)^2}{\varepsilon_{33}^0}L_n^2[\mu_n-(1+\alpha)/2]\right\}\xi^{\mu_n-(3-\alpha)/2}P_n(\cos\theta),\\
\sigma_{r\theta}&=\sigma_{r\theta}^g+\sum_{n=1}^{\infty}A_n^*\left\{c_{44}^0[L_n^1-\mu_n+(3+\alpha)/2]+\frac{e_{15}^0e_{33}^0}{\varepsilon_{33}^0}L_n^2\right\}\xi^{\mu_n-(3-\alpha)/2}\frac{\partial}{\partial\theta}P_n(\cos\theta),\\
\sigma_{\theta\theta}+\sigma_{\phi\phi}&=\sigma_{\theta\theta}^g+\sigma_{\phi\phi}^g+\sum_{n=0}^{\infty}A_n^*\left\{(c_{11}^0+c_{12}^0)[2L_n^1+n(n+1)]+2c_{13}^0L_n^1[\mu_n-(1+\alpha)/2]\right.\\
&\quad\left.+\frac{2e_{31}^0e_{33}^0}{\varepsilon_{33}^0}L_n^2[\mu_n-(1+\alpha)/2]\right\}\xi^{\mu_n-(3-\alpha)/2}P_n(\cos\theta),\\
\sigma_{\theta\theta}-\sigma_{\phi\phi}&=\sigma_{\theta\theta}^g-\sigma_{\phi\phi}^g+\sum_{n=1}^{\infty}A_n^*(c_{11}^0-c_{12}^0)\left[n(n+1)P_n(\cos\theta)+2\cot\theta\frac{\partial}{\partial\theta}P_n(\cos\theta)\right]\xi^{\mu_n-(3-\alpha)/2},\\
D_r&=D_r^g+\sum_{n=0}^{\infty}A_n^*\{e_{33}^0L_n^1[\mu_n-(1+\alpha)/2]+e_{31}^0[2L_n^1+n(n+1)]\\
&\quad-e_{33}^0L_n^2[\mu_n-(1+\alpha)/2]\}\xi^{\mu_n-(3-\alpha)/2}P_n(\cos\theta),\\
D_\theta&=D_\theta^g+\sum_{n=1}^{\infty}A_n^*\left\{e_{15}^0[L_n^1-\mu_n+(3+\alpha)/2]-\frac{\varepsilon_{11}^0e_{33}^0}{\varepsilon_{33}^0}L_n^2\right\}\xi^{\mu_n-(3-\alpha)/2}\frac{\partial}{\partial\theta}P_n(\cos\theta),
\end{aligned}
\tag{49}
$$

where

$$\begin{cases} A_0^* = C_0^*, \quad L_0^1 = B_0^*/C_0^*, \quad L_0^2 = 1, & n = 0, \\ L_n^1 = B_n^*/A_n^*, \quad L_n^2 = C_n^*/A_n^*, & n > 0. \end{cases} \tag{50}$$

It is now supposed that the piezoceramic spherical shell, with the internal and external radii being a and b, respectively, rotate at an angular velocity ω about the polar axis. The centrifugal force acting on the shell may thus be represented as a body force as follows (Chen, 1966):

$$F_r = \frac{2\rho\omega^2 r}{3}[P_0(\cos\theta) - P_2(\cos\theta)], \quad F_\theta = -\frac{\rho\omega^2 r}{3}\frac{\partial}{\partial\theta}P_2(\cos\theta), \quad F_\phi = \rho_{\rm f} = 0. \tag{51}$$

Hence, in Eq. (28),

$$E_0 = -E_2 = -2F_2 = 2R\rho_0\omega^2/3, \quad \mu_0 = \mu_2 = (7+\alpha)/2, \tag{52}$$

with $R = (a+b)/2$. From the free surface boundary conditions, one can derive the linear equations to determine the unknowns A_{ni}. Thus the elasto-electric field of the rotating piezoceramic spherical shell is exactly obtained. As for the general axisymmetric boundary-value problem considered in the last section, we have totally three unknowns, i.e. A_{01}, A_{02} and A_{05} when $n = 0$. Meanwhile, there are four boundary conditions, i.e. $\sigma_{rr} = D_r = 0$ for $r = a, b$. We have demonstrated that the terms corresponding to v_{02} and v_{05} in the expression of D_r vanish everywhere in the shell. That is to say, only the term related to A_{01} does not equal zero. Moreover, since $L_0^1 = B_0^*/C_0^* = (3 + 2f_6)$, we can easily verify that the term corresponding to the particular solution in D_r also vanishes. Thus the two boundary conditions $D_r = 0$ $(r = a, b)$ all result to $A_{01} = 0$. We finally get two equations from the boundary conditions $\sigma_{rr} = 0$ $(r = a, b)$ to determine the other two unknowns A_{02} and A_{05}.

As a numerical example, distributions of the non-dimensional stresses $\sigma_1 = \sigma_{\theta\theta}/(\rho_0\omega^2R^2)$ and $\sigma_2 = \sigma_{\phi\phi}/(\rho_0\omega^2R^2)$, and the non-dimensional electric displacement $D = D_\theta c_{44}/(e_{33}\rho_0\omega^2R)$ along the radial direction are shown in Figs. 1–6. The material is taken to be PZT-4. Dunn and Taya (1994) have listed its material constants for the homogeneous case. It is noted here that for FGMs with a simple power-law distribution as considered in this paper, the non-dimensional material constants $f_1 - f_8$ will be the same as those of the homogeneous ones (Chen et al., 1997; Chen, 1999).

Figs. 1–4 are for a moderately thick shell with the thickness-to-mean radius ratio $(b-a)/R = 0.5$, while Figs. 5 and 6 for a thin shell with the ratio being 0.1. From these figures, it can be shown that the inhomogeneity has a very significant effect on the distributions of stresses and electric displacements in the shell.

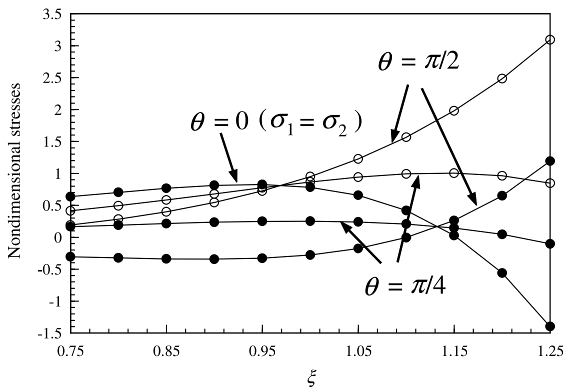

Fig. 1. Distributions of the non-dimensional stresses $\sigma_1 = \sigma_{\theta\theta}/(\rho_0\omega^2R)$ (–●–) and $\sigma_2 = \sigma_{\phi\phi}/(\rho_0\omega^2R)$ (–○–) along the radial direction for $\alpha = 5.0$.

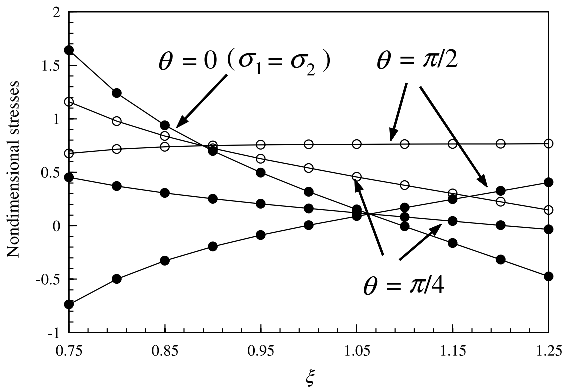

Fig. 2. Distributions of the non-dimensional stresses σ_1 (–●–) and σ_2 (–○–) along the radial direction for $\alpha = 0.0$.

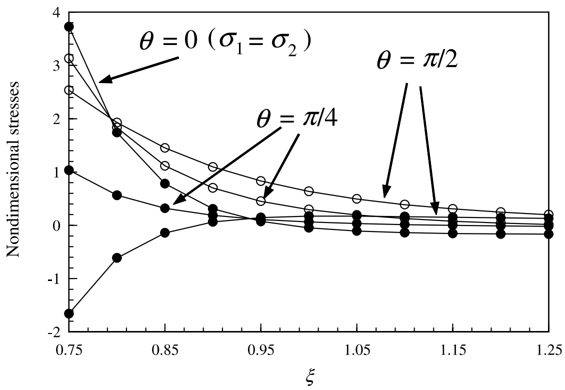

Fig. 3. Distributions of the non-dimensional stresses σ_1 (–●–) and σ_2 (–○–) along the radial direction for $\alpha = -5.0$.

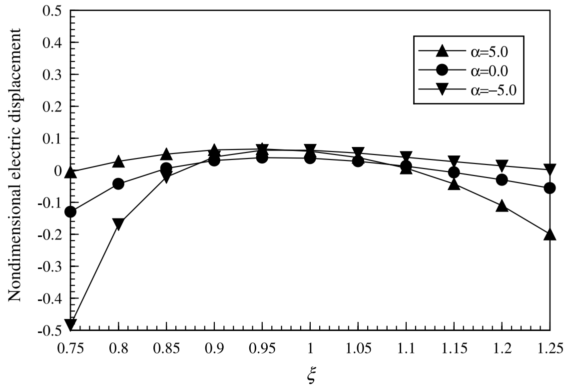

Fig. 4. Distribution of the non-dimensional electric displacement $D = D_\theta c_{44}/(e_{33}\rho_0\omega^2 R)$ along the radial direction when $\theta = \pi/4$.

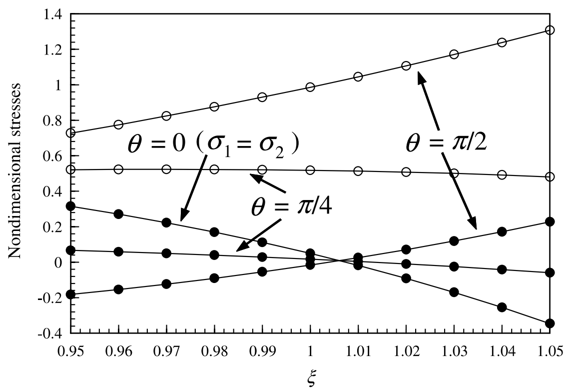

Fig. 5. Distributions of the non-dimensional stresses σ_1 (–●–) and σ_2 (–○–) along the radial direction for a thin spherical shell ($\alpha = 5.0$).

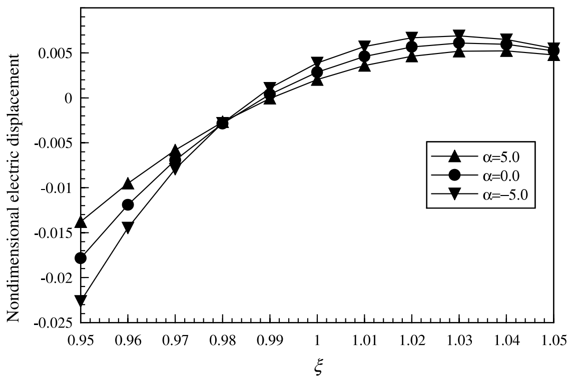

Fig. 6. Distribution of the non-dimensional electric displacement D along the radial direction when $\theta = \pi/4$ for a thin spherical shell.

In fact, adopting a certain value of the inhomogeneity parameter α can optimize not only the stress (electric displacement) level but also the distribution configuration. This will be of particular importance in modern engineering design. From Fig. 5, it can also be seen that, for the thin spherical shell, the distribution of stress σ_1 along the radial direction is nearly linear even for the non-homogeneous case ($\alpha = 5.0$). This agrees with the assumption employed in the classical shell theory. However, the distribution of the electric displacement is far from linear as one can see from Fig. 6. This implies that to develop a two-dimensional approximate theory one should use a high-order mode to fit the electric field even for a thin piezoceramic shell.

6. Conclusions

In this paper, we simplify the basic equations of a spherically isotropic piezoelastic medium with a FGM by using the displacement separation technique. For the particular case that all the material constants are of power functions with an identical exponent of the radial coordinate, the general as well as the particular solutions to the resulted ordinary differential equations are obtained exactly. The elasto-electric field of a rotating piezoceramic spherical shell is then analyzed. In fact, exact expressions for the elasto-electric field are presented. Numerical investigation shows that the inhomogeneous character have an obvious effect on both the stress (electric displacement) level and their distribution configurations.

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